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INTEGRATION OF THE SOLUTION OF A MIXED LING HEAT PROBLEM

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A solution of the Ling heat problem with mixed boundary conditions is obtained using the method of piecewise-linear approximation of the flow. For a constant intensity of the frictional heat flux a resolvent of the kernel of the integral equation of the problem is constructed.

Formulation of the Problem. In [1], the authors proposed a method for integrating the quasistationary problem of heat conduction for a half-space $-\infty < x < \infty$, $0 \le y < \infty$ in the heating of its surface $y = 0$ by a linear distributed heat flux of intensity $q(x) = fVp(x)$, $0 \le x < 2a$. It was assumed that outside the heating portion the surface of the half-space is heat-insulated: $q(x) = 0$, $-\infty < x < 2a \cup 2a < x < \infty$. In this work, we investigate the same problem on condition that the heat exchange of the half-space surface with the ambient medium follows Newton's law. For this we consider a mixed quasistationary problem of heat conduction of the form

$$\frac{\partial^2 T}{\partial \eta^2} = \frac{\partial T}{\partial \xi}\,, \quad -\infty < \xi < \infty\,, \quad 0 \le \eta < \infty\,; \tag{1}$$

$$K \frac{\partial T}{\partial \eta}\bigg|_{\eta=0} = \begin{cases} -\Lambda p^*(\xi)\,, & 0 \le \xi \le 1\,, \\ \mathrm{Bi}\, T\,, & -\infty < \xi < 0 \cup 1 < \xi < \infty\,; \end{cases} \tag{2}$$

$$T \to 0 \quad \text{for} \quad \sqrt{\xi^2 + \eta^2} \to \infty\,. \tag{3}$$

In relations (1)–(3) the notation is as follows:

$$\xi = \frac{x}{2a}\,, \quad \eta = \frac{y}{d}\,, \quad p^* = \frac{p}{p_0}\,,$$
$$d = \sqrt{\frac{2ak}{V}}\,, \quad \mathrm{Bi} = \frac{hd}{K}\,, \quad \Lambda = \frac{fVp_0 d}{K}\,. \tag{4}$$

The form of Eq. (1) indicates that the case of high-speed (Pe > 5) heating of the surface of the half-space is considered.

The solution of boundary-value problem (1)–(3) obtained using the Fourier integral transformation has the form [2]

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$$T(\xi, \eta) = \begin{cases} 0\,, & -\infty < \xi \le 0\,, \\ \dfrac{\Lambda}{\sqrt{\pi}} \displaystyle\int_0^{\xi} G(\xi - \tau, \eta)\, p^*(\tau)\, d\tau\,, & 0 \le \xi \le 1\,, \\ \dfrac{\Lambda}{\sqrt{\pi}} \displaystyle\int_0^{1} G(\xi - \tau, \eta)\, p^*(\tau)\, d\tau - \dfrac{\mathrm{Bi}}{\sqrt{\pi}} \displaystyle\int_1^{\xi} G(\xi - \lambda, \eta)\, T(\lambda)\, d\lambda\,, & 1 \le \xi < \infty\,, \end{cases} \tag{5}$$

where

$$G(\xi, \eta) = \frac{1}{\sqrt{\xi}} \exp\left(-\frac{\eta^2}{4\xi}\right), \quad 0 \le \eta < \infty\,; \tag{6}$$

$T(\lambda) \equiv T(\lambda, 0)$ is the temperature of the half-space surface.

The solution (5) and (6) differs from the corresponding solution in the case of heat insulation of the free surface of the half-space [1] only by the presence of a term with the factor $\mathrm{Bi}/\sqrt{\pi}$ in the relation for the temperature in the region $1 \le \xi < \infty$. However, this complicates substantially the calculation of the temperature field $T(\xi, \eta)$ since first we must find the surface temperature $T(\xi)$, $-\infty < \xi < \infty$.

Analysis of the Solution (6). A method for determining the temperature field $T(\xi, \eta)$ for $0 \le \xi \le 1$ and $0 \le \eta < \infty$ by formulas (5) and (6) is presented in [1]. But direct employment of it in the region $1 \le \xi < \infty$, $0 \le \eta < \infty$ behind the heating portion requires some preparatory work. In particular, from formulas (5) and (6) it follows that for $\eta = 0$ we must solve the Volterra integral equation of the second kind with a weakly singular kernel

$$T(\xi) + \frac{\mathrm{Bi}}{\sqrt{\pi}} \int_1^{\xi} \frac{T(\lambda)}{\sqrt{\xi - \lambda}} = F(\xi)\,, \quad 1 \le \xi < \infty\,, \tag{7}$$

where

$$F(\xi) = \frac{\Lambda}{\sqrt{\pi}} \int_0^{1} \frac{p^*(\tau)\, d\tau}{\sqrt{\xi - \tau}}\,. \tag{8}$$

The resolvent of the kernel of integral equation (7) has the form [3]

$$R(\xi) = \sum_{n=1}^{\infty} \frac{(-\mathrm{Bi}\sqrt{\xi})^n}{\xi \Gamma\left(\dfrac{n}{2}\right)}\,. \tag{9}$$

For $n = 2k$, from relation (9) we find

$$R_{2k}(\xi) = \sum_{k=1}^{\infty} \frac{(\mathrm{Bi}^2 \xi)^k}{\xi \Gamma(k)} = \mathrm{Bi}^2 \sum_{k=1}^{\infty} \frac{(\mathrm{Bi}^2 \xi)^{k-1}}{(k-1)\,!} = \mathrm{Bi}^2 \exp(\mathrm{Bi}^2 \xi)\,, \tag{10}$$

and for $n = 2k + 1$, having used formula 5.2.7.18 of [4], we have

$$R_{2k+1}(\xi) = \sum_{k=0}^{\infty} \frac{(-\mathrm{Bi}\sqrt{\xi})^{2k+1}}{\xi\Gamma\left(k+\frac{1}{2}\right)} = -\frac{\mathrm{Bi}\sqrt{\xi}}{\xi\Gamma\left(\frac{1}{2}\right)} - \sum_{k=1}^{\infty} \frac{(\mathrm{Bi}\sqrt{\xi})^{2k+1}}{\xi\Gamma\left(k+\frac{1}{2}\right)} =$$

$$= -\frac{\mathrm{Bi}}{\sqrt{\pi\xi}} - \mathrm{Bi}^2 \sum_{k=0}^{\infty} \frac{(\mathrm{Bi}\sqrt{\xi})^{2k+1}}{\Gamma\left(k+\frac{3}{2}\right)} = -\frac{\mathrm{Bi}}{\sqrt{\pi\xi}} - \mathrm{Bi}^2 \exp(\mathrm{Bi}^2\xi)\,\mathrm{erf}(\mathrm{Bi}\sqrt{\xi}) \,. \tag{11}$$

By summing relations (10) and (11) we obtain

$$R(\xi) = R_{2k}(\xi) + R_{2k+1}(\xi) = -\frac{\mathrm{Bi}}{\sqrt{\pi\xi}} + \mathrm{Bi}^2 \exp(\mathrm{Bi}^2\xi)\,\mathrm{erfc}(\mathrm{Bi}\sqrt{\xi}) \,. \tag{12}$$

The resolvent $R(\xi)$ of (12) allows the asymptotic form

$$R(\xi) \cong \tilde{R}(\xi) = -\frac{\mathrm{Bi}}{\sqrt{\pi\xi}} + \mathrm{Bi}^2\left(1 - 2\,\mathrm{Bi}\sqrt{\frac{\xi}{\pi}} + \mathrm{Bi}^2\xi\right) \quad \text{for} \quad \mathrm{Bi}\sqrt{\xi} << 1 \,, \tag{13}$$

$$R(\xi) \cong \tilde{\tilde{R}}(\xi) = -\frac{1}{2\mathrm{Bi}\,\xi\sqrt{\pi\xi}} \quad \text{for} \quad \mathrm{Bi}\sqrt{\xi} >> 1 \,. \tag{14}$$

Knowing the resolvent $R(\xi)$ of (12), we write the solution of the Volterra integral equation (7) in the form

$$T(\xi) = F(\xi) + \int_1^{\xi} R(\xi-\tau)\,F(\tau)\,d\tau \,, \quad 1 \le \xi < \infty \,, \tag{15}$$

where the function $F(\xi)$ is determined by formula (8). Thus, the problem has been reduced to evaluation of the integral in the right-hand side of equality (15). Clearly it is more appealing to use for this purpose the asymptotic forms (13) and (14) since they contain no probability integrals.

A numerical investigation of the behavior of the resolvent $R(\xi-\tau)$ showed that its approximation $\tilde{R}(\xi-\tau)$ using formula (13) is rather (with an absolute error lower than 1%) accurate for values of the argument $\xi - \tau \le \mathrm{Bi}^{-2}$. Hence it follows that the integral

$$\int_1^{\xi} \tilde{R}(\xi-\tau)\,F(\tau)\,d\tau \,, \quad 1 \le \xi < \infty \,, \tag{16}$$

can be evaluated for $1 \le \xi \le 1 + \mathrm{Bi}^{-2}$. We find the range of variation of the Biot number for wheel–rail tribosystems. For steel, K = 41 W/(m·K) and $k = 9.1 \cdot 10^{-6}$ m^2/sec. In the case of frictional heat generation in sliding of a locomotive wheel over a rail the width of the heating area is $2a \approx 10^{-2}$ m and the heat-transfer coefficient does not exceed $h \le 200$ W/(m^2·K) [5]. Taking into account notation (4), we obtain the estimate $\mathrm{Bi} \le 0.15 \cdot 10^{-2}/\sqrt{V}$ for the Biot number. The maximum sliding velocity for the dynamic load of the locomotive wheel is $V \approx 0.75$ m/sec. Thus, we find that the upper limit of the change in the Biot number for the wheel–rail tribocontact does not exceed 10^{-2}. For heat generation on the actual portions of contact,

$2a \approx 10^{-4}$ m [6], and then this limit will decrease by another order of magnitude. Therefore the integral (16) can certainly be calculated for $1 \le \xi \le 10^4$.

Uniform Distribution of Contact Pressure. We investigate solution (5) and (6) in the case of a constant intensity of the pressure

$$p(x) = \frac{\pi}{4} p_0 , \quad 0 \le x \le 2a , \tag{17}$$

where $p_0 = 2P/(\pi a)$ is the maximum value of the pressure for an elliptical (Hertz) distribution [7]. Substitution of the function $p^*(\tau) = \pi/4$ of (17) under the integrals in relations (5) and integration using results of [1] yield

$$T(\xi, \eta) = \begin{cases} 0 , & -\infty < \xi \le 0 , \\ T_{\max}\theta(\xi, \eta) , & 0 \le \xi \le 1 , \\ T_{\max}[\theta(\xi, \eta) - \theta(\xi - 1, \eta)] - \dfrac{\mathrm{Bi}}{\sqrt{\pi}} \displaystyle\int_1^{\xi} G(x - \lambda, \eta)\, T(\lambda)\, d\lambda , & 1 \le \xi < \infty , \end{cases} \tag{18}$$

where

$$\theta(\xi, \eta) = \sqrt{\xi} \exp\left(-\frac{\eta^2}{4\xi}\right) - \eta \frac{\sqrt{\pi}}{2} \operatorname{erfc}\left(\frac{\eta}{2\sqrt{\xi}}\right), \quad 0 \le \eta < \infty , \tag{19}$$

$T_{\max} = \Lambda\sqrt{\pi/2}$ is the maximum value of the temperature, which is attained at the point $\xi = 1$ on the half-space surface [1].

The surface temperature of the half-space behind the heating portion of determined from Eq. (15), where with account for relation (19)

$$F(\xi) = T_{\max}(\sqrt{\xi} - \sqrt{\xi - 1}) , \quad 1 \le \xi \le \infty . \tag{20}$$

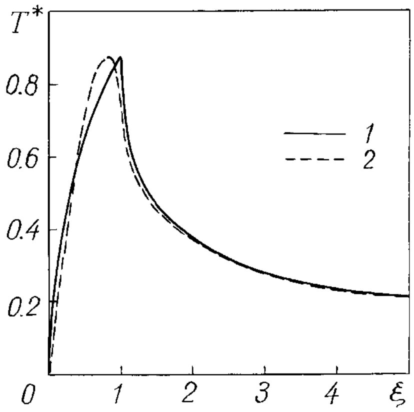
Having substituted the resolvent $\tilde{R}(\xi)$ of (13) into solution (15) and having integrated, we obtain, with account for relation (20), a formula for determining the temperature of the part of the surface of the half-space behind the heating strip:

$$T(\xi) = T_{\max} T^*(\xi) , \quad T^*(\xi) = \sqrt{\xi} - \sqrt{\xi - 1} - \mathrm{Bi}\, \theta_1(\xi) , \quad 1 \le \xi \le \infty , \tag{21}$$

where

$$\theta_1(\xi) = \frac{1}{\sqrt{\pi}}\left[\sqrt{\xi - 1} - \frac{\pi}{2}(\xi - 1) + \xi \arcsin\sqrt{1 - \frac{1}{\xi}}\right] - \frac{2}{3}\mathrm{Bi}\,[\xi\sqrt{\xi} -$$

$$- 1 - (\xi - 1)\sqrt{\xi - 1}] + \frac{\mathrm{Bi}^2}{2\sqrt{\pi}}\left[(\xi - 2)\sqrt{\xi - 1} - \frac{\pi}{2}(\xi - 1)^2 + \right.$$



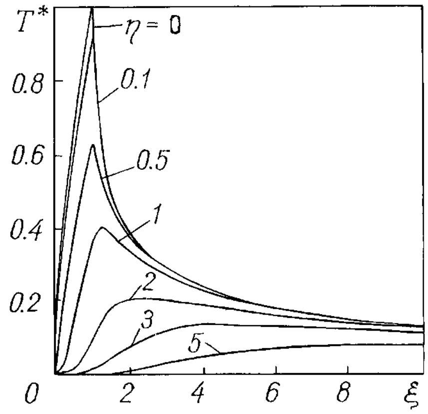


Fig. 1. Distribution of the dimensionless temperature T^* of the half-space surface for different values of the Biot number for a constant intensity of the contact pressure (17).

Fig. 2. Distribution of the dimensionless temperature T^* at different distances η from the half-space surface for Bi = 0.05 for a constant intensity of the contact pressure (17).

$$\left. + \xi^2 \arcsin \sqrt{1 - \frac{1}{\xi}} \right] - \frac{2}{15} \mathrm{Bi}^3 \left[3 - 5\xi + 2\xi^2 \sqrt{\xi} - 2(\xi - 1)^2 \sqrt{\xi - 1}\right] . \tag{22}$$

Knowing the surface temperature, from formulas (18)–(22) we can calculate the temperature field at an arbitrary point of the half-space. Here it is necessary to determine the integral of the product of the functions $G(\xi - \lambda, \eta)$ of (6) and $T(\lambda)$ of (21). For this purpose we employ the procedure of approximation using piecewise-linear functions [8]. We represent the function $T^*(\lambda)$ in the form

$$T^*(\lambda) \cong \tilde{T}^*(\lambda) = \sum_{j=0}^{m} T_j^* \varphi_j(\lambda) , \quad T_j^* = T^*(\lambda_j) , \tag{23}$$

where $\varphi_j(\lambda)$ are piecewise-linear functions, $\lambda_j = 1 + j\delta\lambda$, $j = 0, 1, ..., m$, and $\delta\lambda = (\xi - 1)/m$.

Substitution of approximation (23) into relations (18) yields a formula for calculating the temperature field in the half-space behind the heating portion:

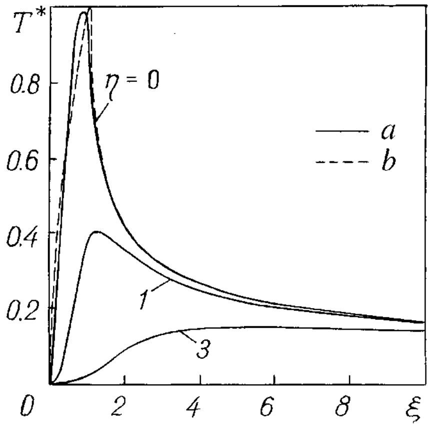
$$T(\xi, \eta) = T_{\max} T^*(\xi, \eta) , \quad 1 \le \xi < \infty , \quad 0 \le \eta < \infty ;$$

$$T^*(\xi, \eta) = \theta(\xi, \eta) - \theta(\xi - 1, \eta) - \frac{\mathrm{Bi}}{\sqrt{\pi}\,\delta\lambda} \sum_{j=0}^{m} G_j^{(1)}(\xi, \eta)\, T_j^* , \tag{24}$$

the form of the functions $G_j^{(1)}(\xi, \eta)$ is given in [1].

The dimensionless temperature T^* (21) and (22) of the surface of the half-space behind the heating strip decreases with increase in the Biot number (Fig. 1).

For a fixed value of Bi the dimensionless temperature T^* of (24) decreases rapidly with distance from the half-space surface (Fig. 2). This process is most rapid under the heating zone $0 \le \xi \le 1$. In particular, the effective depth of heating is approximately $3d$ in the cross section $\xi = 1$ and about $3d$ in the cross section $\xi = 2$, while it increases to $7d$ for $\xi = 5$.



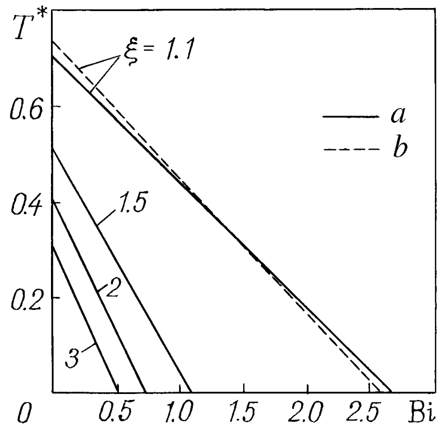


Fig. 3. Distribution of the dimensionless temperature T^* at different distances η from the surface of a half-space for Bi = 0.01: a) Hertz distribution of the pressure (28); b) constant pressure (17).

Fig. 4. Dimensionless temperature T^* of the half-space surface vs. Biot number in different cross sections ξ: a, b) the notation is the same as in Fig. 3.

Arbitrary Distribution of Contact Pressure. If the function $p^*(\tau)$ is not constant, then, to find the temperature $T(\xi, \eta)$ from formulas (5) we use the expansion

$$p^*(\tau) \cong \tilde{p}^*(\tau) = \sum_{i=0}^{n} p_i^* \varphi_i(\tau), \quad p_i^* = p^*(\tau_i), \tag{25}$$

where $\tau_i = i\delta\tau$, $i = 0, 1, 2, ..., n$ and $\delta\tau = 1/n$.

Substitution of approximation formulas (23) and (25) into relations (5) for $1 \le \xi < \infty$ and $\eta = 0$ leads to the triangular system of linear algebraic equations

$$T_k^* + \frac{\mathrm{Bi}}{\sqrt{\pi}\,\delta\lambda} \sum_{j=0}^{k} G_{jk}^{(1)} T_j^* = \frac{2}{\pi\delta\tau} \sum_{i=0}^{n} p_i^* G_{ik}^{(1)}, \quad k = 0, 1, 2, ..., m, \tag{26}$$

where $G_{jk}^{(1)} = G_j^{(1)}(\lambda_k, 0)$.

As a result of solving the system of equations (26) we find the values of the dimensionless surface temperature at a discrete set of points $T_j^* = T^*(\lambda_j)$, $j = 0, 1, ..., m$. Then from formulas (5) we find the temperature distribution in the half-space

$$T(\xi, \eta) = T_{\max} T^*(\xi, \eta), \quad 1 \le \xi < \infty, \quad 0 \le \eta < \infty,$$

$$T^*(\xi, \eta) = \frac{2}{\pi\delta\tau} \sum_{i=0}^{n} G_i^{(1)}(\xi, \eta)\, p_i^* - \frac{\mathrm{Bi}}{\sqrt{\pi}\,\delta\lambda} \sum_{j=0}^{m} G_j^{(1)}(\xi, \eta)\, T_j^*. \tag{27}$$

Finally we note that the expressions for finding the temperature in the region $0 \le \xi \le 1$, $0 < \eta < \infty$ are the same as in the case of heat insulation of the free surface of the half-space [1].

One most frequently uses the Hertz distribution of the contact pressure

$$p(x) = p_0 p^*(x) \,, \quad p^*(x) = \sqrt{1 - \left(\frac{x-a}{a}\right)^2} \,, \quad 0 \le x \le 2a \,. \tag{28}$$

Computations by formulas (26) and (27) showed that a noticeable difference in the temperature distribution in using formulas (17) and (28) is observed only in the heating strip $0 \le \xi \le 1$ (Fig. 3). Behind this region for $1 \le \xi \le \infty$ and $0 \le \eta \le \infty$ the corresponding temperatures practically coincide. Hence, with the Hertz pressure distribution (28), too, one can use the analytical solution (21) for calculating the surface temperature behind the frictional heat source!

The dependence of the surface temperature T^* on the Biot number turned out to be linear (Fig. 4). A slight difference in these dependences for the constant (17) and Hertz (28) distributions of the contact pressure is observed only in the immediate vicinity ($1 \le \xi \le 1.1$) of the heating region.

NOTATION

T, temperature; $T^*(\xi, \eta) = T(\xi, \eta)/\Lambda$, dimensionless temperature; Λ, constant determined from formula (4) and having the dimensions of temperature; h, heat-transfer coefficient; a, halfwidth of the heating region; K, thermal-conductivity coefficient; k, thermal-diffusivity coefficient; V, velocity of motion of the linear heat flux; $\Gamma(\cdot)$, gamma function; $q(x)$, intensity of the heat flux; (x, y), axes of the orthogonal Euler coordinate system; f, coefficient of friction; p, contact pressure; p_0, characteristic value of the contact pressure; d, effective depth of heating; Pe, Péclet parameter; Bi, Biot number; P, linear pressing force; $H(\cdot)$, Heaviside unit function; erf(·), probability integral; erfc(·) = 1 – erf(·).

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